# COMPLETE GRADIENT SHRINKING RICCI SOLITONS HAVE FINITE TOPOLOGICAL TYPE

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Abstract. We show that a complete Riemannian manifold has finite topological type (i.e., homeomorphic to the interior of a compact manifold with boundary), provided its Bakry-Émery Ricci tensor has a positive lower bound, and either of the following conditions:

(i) the Ricci curvature is bounded from above;
(ii) the Ricci curvature is bounded from below and injectivity radius is bounded away from zero.

Moreover, a complete shrinking Ricci soliton has finite topological type if its scalar curvature is bounded.

## 1. Introduction

In 1968, J. Milnor [8] conjectured that a complete non-compact Riemannian manifold with non-negative Ricci curvature has a finitely generated fundamental group. However, such a manifold may not have finite topological type. Examples of complete non-compact manifold with positive Ricci curvature without finite topological type was constructed by Gromoll-Meyer [5]. It has been an interesting topic in Riemannian geometry to study the topology of complete manifolds with positive (non-negative) Ricci curvature.

In this note we are concerned with complete Riemannian manifold $(M, g)$ satisfying that $\mathrm{Ric}+\mathrm{Hess}(f) \geq \lambda g$ for some constant $\lambda > 0$ and $f \in C^\infty(M)$, i.e., whose Bakry-Émery Ricci tensor is bounded below by $\lambda$ in the sense of [7]. When the equality holds, the manifold is a shrinking Ricci soliton, i.e., a self-similar solution of the well-known Ricci flow equation. If $f$ is constant, Bakry-Émery Ricci tensor reduces to the Ricci tensor, and so the classical Myers' theorem implies that $M$ is compact with finite fundamental group. In general, $M$ may not be compact, but from the work of [4, 6, 7, 9, 10, 11] etc., $M$ still has finite fundamental group.

The main result of this note shows that a complete Riemannian manifold whose Bakry-Émery Ricci tensor is bounded below by $\lambda > 0$ has finite topological type, provided the Ricci curvature is bounded from above. Moreover,

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a shrinking Ricci soliton has finite topological type if its scalar curvature is bounded.

**Theorem 1.** *Suppose $(M, g)$ is a complete Riemannian manifold satisfying $Ric + Hess(f) \geq \lambda g$ for some constant $\lambda > 0$ and $f \in C^\infty(M)$. Then $M$ is of finite topological type, if either of the following alternative conditions holds:*

(i) *$Ric \leq Cg$ for some constant $C < \infty$;*
(ii) *$Ric \geq -\delta^{-1} g$ and the injectivity radius $\mathrm{inj}(M, g) \geq \delta > 0$ for some $\delta > 0$.*

If $(M, g)$ is a shrinking Ricci soliton, then the Ricci curvature bounds can be relaxed by scalar curvature.

**Theorem 2.** *Suppose $(M, g)$ is a complete shrinking Ricci soliton $Ric + Hess(f) = \frac{g}{2}$, where $f \in C^\infty(M)$. If the scalar curvature $R$ is bounded, then $M$ has finite topological type.*

In view of Theorem 2 it is nature to pose the following

**Conjecture 3.** *Any shrinking Ricci soliton has finite topological type.*

We prove Theorem 1 in section 2 and Theorem 2 in section 3.

## 2. Proof of theorem 1

Let $(M, g)$ be such a manifold satisfying that $\mathrm{Ric} + \mathrm{Hess}(f) \geq \lambda g$ for some $\lambda > 0$ and $f \in C^\infty(M)$. By the deformation lemma of Morse theory, to prove Theorem 1, it suffices to show that the function $f$ is proper and has no critical points outside of a compact set.

First fix one point $p \in M$ as a base point. For any $q \in M$ with $d(p, q) = L$, choose a shortest geodesic $\gamma$ from $p$ to $q$ parametrized by arc length. Then

$$\begin{aligned} \langle \nabla f, \dot\gamma \rangle(q) &= \langle \nabla f, \dot\gamma \rangle(p) + \int_0^L \frac{d^2}{dt^2} f(\gamma(t))dt \\ &\geq \langle \nabla f, \dot\gamma \rangle(p) + \int_0^L (\lambda - Ric(\dot\gamma, \dot\gamma))dt \\ &\geq \lambda L - |\nabla f|(p) - \int_0^L Ric(\dot\gamma, \dot\gamma)dt. \end{aligned}$$

If the integral

$$\int_0^L Ric(\dot\gamma, \dot\gamma)dt \leq \Lambda$$

for some constant $\Lambda$ independent of $q$ and the choice of $\gamma$, then

$$|\nabla f|(q) \geq \langle \nabla f, \dot\gamma \rangle(q) \geq \lambda d(p, q) - |\nabla f|(p) - \Lambda,$$

which implies that $|\nabla f|(q)$ has a linear growth in $d(p,q)$ and so $f$ is a proper function without critical points outside of a compact set. In the remainder of this section, we will focus on proving that $\int_0^L Ric(\dot\gamma,\dot\gamma)$ has an upper bound under the assumptions of Theorem 1.

Case (i): $\mathrm{Ric} \le Cg$ for some constant $C < \infty$;
By Lemma 2.2 of [9], the integral bound is given by $\Lambda = 2(n-1) + 2C$.

Case (ii): $\mathrm{Ric} \ge -\delta^{-1} g$ and $\mathrm{inj}(M,g) \ge \delta > 0$ for some $\delta > 0$.
Suppose $d(p,q) = L \ge \delta$. Let $\varphi(t) : [0,L] \to [0,1]$ be an arcwise smooth function such that $\varphi(0) = \varphi(L) = 0$. By the second variation formula, as did in [9] or [11], we have the following estimate:

$$\int_0^L \varphi^2(t) Ric(\dot\gamma,\dot\gamma)dt \le (n-1)\int_0^L |\dot\varphi|^2 dt.$$

Now define $\varphi$ by

$$\varphi = \begin{cases} \frac{3}{\delta}t, & t \in [0,\frac{\delta}{3}]; \\ 1, & t \in [\frac{\delta}{3}, L-\frac{\delta}{3}]; \\ \frac{3}{\delta}(L-t), & t \in [L-\frac{\delta}{3}, L], \end{cases}$$

then we have the estimate

$$\begin{aligned} \int_0^L Ric(\dot\gamma,\dot\gamma)dt &\le (n-1)\int_0^L |\dot\varphi|^2 dt + \int_0^{\frac{\delta}{3}} (1-\varphi^2) Ric(\dot\gamma,\dot\gamma)dt + \int_{L-\frac{\delta}{3}}^L (1-\varphi^2) Ric(\dot\gamma,\dot\gamma)dt \\ &\le \frac{6}{\delta}(n-1) + \frac{2}{3} + \int_0^{\frac{\delta}{3}} Ric(\dot\gamma,\dot\gamma)dt + \int_{L-\frac{\delta}{3}}^L Ric(\dot\gamma,\dot\gamma)dt, \end{aligned}$$

where in the second inequality, we used the fact that

$$\begin{aligned} \int_0^{\frac{\delta}{3}} (1-\varphi^2) Ric(\dot\gamma,\dot\gamma)dt &= \int_0^{\frac{\delta}{3}} (1-\varphi^2)(Ric(\dot\gamma,\dot\gamma) + \frac{1}{\delta})dt - \frac{1}{\delta}\int_0^{\frac{\delta}{3}} (1-\varphi^2)dt \\ &\le \int_0^{\frac{\delta}{3}} (Ric(\dot\gamma,\dot\gamma) + \frac{1}{\delta})dt - \frac{1}{\delta}\int_0^{\frac{\delta}{3}} (1-\varphi^2)dt \\ &\le \int_0^{\frac{\delta}{3}} Ric(\dot\gamma,\dot\gamma)dt + \frac{1}{\delta}\int_0^{\frac{\delta}{3}} \varphi^2 dt \\ &\le \int_0^{\frac{\delta}{3}} Ric(\dot\gamma,\dot\gamma)dt + \frac{1}{3}, \end{aligned}$$

and similarly

$$\int_{L-\frac{\delta}{3}}^L (1-\varphi^2) Ric(\dot\gamma,\dot\gamma)dt \le \int_{L-\frac{\delta}{3}}^L Ric(\dot\gamma,\dot\gamma)dt + \frac{\delta}{3}.$$

We next prove that $\int_0^{\frac{\delta}{3}} Ric(\dot\gamma,\dot\gamma)dt$ and $\int_{L-\frac{\delta}{3}}^{L} Ric(\dot\gamma,\dot\gamma)dt$ are bounded from above and so finish the proof of Theorem 1. This is given by the following lemma.

**Lemma 4.** *If $Ric \geq -\delta^{-1}g$ and $\operatorname{inj}(M,g) \geq \delta > 0$ for some $\delta > 0$, then*

$$\int_0^{\frac{\delta}{3}} Ric(\dot\gamma,\dot\gamma)dt \leq \frac{6}{\delta}(n-1) + \frac{2}{3}$$

*for any minimal arc length parametrized geodesic $\gamma : [0,\frac{\delta}{3}] \to M$.*

*Proof.* Firstly, by $\operatorname{inj}(M,g) \geq \delta$, we can extend the geodesic $\gamma$ to a shortest geodesic $\sigma : [0,\delta] \to M$, such that $\gamma(t) = \sigma(t+\frac{\delta}{3}), t \in [0,\frac{\delta}{3}]$.

Set $L = \delta$ in the arguments above, we have

$$\int_0^{\delta} \varphi^2 Ric(\dot\sigma,\dot\sigma)dt \leq (n-1)\int_0^{\delta} |\dot\varphi|^2 dt = \frac{6}{\delta}(n-1),$$

then using $Ric \geq -\delta^{-1}g$, we get the estimate

$$\begin{aligned}
\int_0^{\frac{\delta}{3}} Ric(\dot\gamma,\dot\gamma)dt &= \int_{\frac{\delta}{3}}^{\frac{2\delta}{3}} Ric(\dot\sigma,\dot\sigma)dt \\
&\leq \frac{6}{\delta}(n-1) - \int_0^{\frac{\delta}{3}} \varphi^2 Ric(\dot\sigma,\dot\sigma)dt - \int_{\frac{2\delta}{3}}^{\delta} \varphi^2 Ric(\dot\sigma,\dot\sigma)dt \\
&\leq \frac{6}{\delta}(n-1) + \frac{2}{3}.
\end{aligned}$$

This concludes the result. $\square$

## 3. Proof of Theorem 2

As before, we will prove that the potential function $f$ to the Ricci soliton is proper and has no critical points outside of $B(p,\rho)$ for large $\rho$.

Suppose $(M,g)$ is a complete shrinking Ricci soliton which satisfies

$$\mathrm{Ric} + \mathrm{Hess}(f) = \frac{g}{2} \tag{1}$$

for some potential function $f$. Suppose further that the scalar curvature $|R| \leq C$ for some constant $C < \infty$. It's well-known that the following analytic equality holds for the soliton ( after modifying $f$ by a translation, see [3] for example.):

$$R + |\nabla f|^2 = f. \tag{2}$$

We begin with several lemmas. Let $q \in M$ be one critical point of $f$ and denote by $\rho = d(p,q)$ the distance from $p$ to $q$. Let $\gamma$ be a shortest arc length parametrized geodesic from $p$ to $q$. Then we have

**Lemma 5.**

$$\frac{\rho}{2} - |\nabla f|(p) \leq \int_0^\rho Ric(\dot{\gamma}, \dot{\gamma})dt. \tag{3}$$

*Proof.* By a direct computation,

$$\begin{aligned} 0 = \langle \nabla f, \dot{\gamma}\rangle(q) &= \langle \nabla f, \dot{\gamma}\rangle(p) + \int_0^\rho \frac{d^2}{dt^2} f(\gamma(t))dt \\ &\geq -|\nabla f|(p) + \int_0^\rho (\frac{1}{2} - Ric(\dot{\gamma}, \dot{\gamma}))dt. \end{aligned}$$

Then the result follows. □

On the other hand, by second variation formula as did in above section, we can get an upper bound for $\int_0^\rho Ric(\dot{\gamma}, \dot{\gamma})dt$. Precisely, for the function $\psi$ defined by

$$\psi(t) = t, t \in [0,1]; \psi(t) \equiv 1, t \in [1, \rho_i - 1]; \psi(t) = \rho_i - t, t \in [\rho_i - 1, \rho_i],$$

we have the estimate

$$\begin{aligned} \int_0^\rho Ric(\dot{\gamma}, \dot{\gamma})dt &\leq \int_0^\rho (n-1)|\dot{\psi}|^2 dt + \int_0^1 (1-\psi^2) Ric(\dot{\gamma}, \dot{\gamma})dt \\ &+ \int_{\rho-1}^\rho (1-\psi^2) Ric(\dot{\gamma}, \dot{\gamma})dt \\ &\leq 2(n-1) + \sup_{B(p,1)} |Ric| + \int_{\rho-1}^\rho (1-\psi^2)(\frac{1}{2} - \frac{d^2}{dt^2} f(\gamma(t)))dt \\ &\leq 2(n-1) + 1 + \sup_{B(p,1)} |Ric| - \int_{\rho-1}^\rho (1-\psi^2) \frac{d^2}{dt^2} f(\gamma(t))dt. \end{aligned}$$

Do integration by parts, we have the estimate for the last term

$$\begin{aligned} -\int_{\rho-1}^\rho (1-\psi^2)\frac{d^2}{dt^2} f(\gamma(t))dt &= 2\int_{\rho-1}^\rho \psi \frac{d}{dt} f(\gamma(t))dt \\ &= -2f(\gamma(\rho-1)) + 2\int_{\rho-1}^\rho f(\gamma(t))dt. \end{aligned}$$

Substituting this equality into above estimate, we obtain

**Lemma 6.**

$$\begin{aligned} \int_0^\rho Ric(\dot{\gamma}, \dot{\gamma})dt &\leq 2n + \sup_{B(p,1)} |Ric| + 2\int_{\rho-1}^\rho f(\gamma(t))dt - 2f(\gamma(\rho-1)) \\ &\leq 2n + \sup_{B(p,1)} |Ric| + \sup_{x,y \in B(q,1)} 2|f(x) - f(y)|. \end{aligned}$$

Now we use equation (2) to give an upper bound of $\int_0^\rho Ric(\dot\gamma,\dot\gamma)dt$. First by equation (2) we have the gradient estimate $|\nabla f| \le \sqrt{f-R} \le \sqrt{f+C}$. Then by assumption, $q$ is a critical point of $f$, so $|f(q)| = |R(q)| \le C$. Integrating along a geodesic, we see that for any $x \in B(q,1)$

$$\sqrt{f(x)+C} \le \sqrt{f(q)+C} + \frac{d(x,q)}{2} \le \sqrt{2C}+1. \tag{4}$$

Thus $f(x) \le 3C+2$ for all $x \in B(q,1)$ and consequently

$$\sup_{x,y\in B(q,1)} |f(x)-f(y)| \le (3C+2+C) = 4C+2. \tag{5}$$

The combination of Lemma 5, Lemma 6 and equation (5) gives the upper bound of the distance $\rho = d(p,q)$:

$$\rho \le 4n + 8C + 4 + 2|\nabla f|(p) + \sup_{B(p,1)} 2|Ric|.$$

Note that the arguments above just used the upper boundedness of $f$. By the same reason as before, we conclude that $f$ is proper and then finish the proof of the theorem.

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